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Some Einstein spaces and their global properties

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Abstract. The global structure of a class of Einstein spaces is investigated. These spaces have algebraically special Weyl tensors and contain homogeneous hypersurfaces. It is found that they display simple examples of a variety of interesting configurations involving horizons and singularities.

1. Introduction

Siklos (1978) has described a method of investigating spatially homogeneous space–times by means of the Newman–Penrose formalism. The object then was to examine solutions of the Einstein field equations in which the space-like surfaces of homogeneity tilt over, becoming null, then time-like (i.e. 'whimper' solutions (Ellis and King 1974)), and the choice of a null invariant tetrad was found to be particularly suitable for this purpose.

In this paper the same method is used to study the existence and global properties of a class of exact solutions. The class consists of Einstein spaces

$$R_{ab} = \tfrac{1}{4}Rg_{ab},$$

for which the Weyl tensor is algebraically special, and which admit homogeneous hypersurfaces.

The null tetrad formalism is described in § 2 and some of the relevant equations are integrated in § 3. This method does not apply to all the solutions in the class. Not covered are solutions for which the hypersurfaces of homogeneity (i) contain all the repeated principal null congruences of the Weyl tensor, or (ii) are all null, or (iii) do not admit a simply transitive group of motions. It is not necessary for present purposes to find these solutions explicitly, because their global properties are straightforward. However, the exact solutions in these exceptional cases will be dealt with in a later paper (MacCallum and Siklos 1980, in preparation).

In § 4 the exact solutions of the equations given in § 3 are presented and their global properties are discussed in § 5. These are summarised in table 1. It emerges that they provide examples of many of the different singularity and horizon configurations described by Ellis and King (1974). Furthermore, two families of solutions are of particular interest; by comparing the spin coefficients given in equation (4.1) and in table (3) with those given in Siklos (1978), one can identify these solutions as two of the

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Table 1. The solutions. See § 4 and § 5 for explanation and references. The +, −, and ± in the fourth column indicates whether the G_3 under consideration acts on space-like, time-like, or space-like and time-like hypersurfaces, respectively.

Petrov type	Group type	Λ	No of independent Killing vectors	Description of solution	Metric eqn.	Global features
{4}	IV, VI_h or VII_h	0	6	Homogeneous plane waves	(4.3) (4.7)	Non-scalar singularity + horizon ('whimper')
{4}	$VI_{-1/9}$	<0	5± OR 5−	Kaigorodov's solution	(4.8)	Two horizons, + non-scalar singularities ('whimper-whimper') OR singularity-free
{4}	$VI_{-1/9}$	<0	3−	Leroy's solution	(4.12)	Singularity-free
{31}	$VI_{-1/9}$	any	3± OR 3+	Generalised Collinson–French	(4.10)	Whimper OR Non-scalar singularity with no horizon
{31}	$VI_{-1/9}$	<0	4−	Kaigorodov's solution	(4.9)	Singularity free
{31}	$VI_{-1/9}$	<0	3−	$\rho \neq \bar{\rho}$	(4.13)	Singularity free
{211}	$VI_{-1/9}$	<0	3−	$\rho = 0$	(4.11)	Singularity free
{22}	I	any	4±	LRS Kasner with cosmological constant	—	Scalar singularity
{22}	II, III, VIII, IX	any	4±	NUT–de Sitter	(4.14)	Many bifurcate horizons, no singularities
{22}	VIII, VI_0 III, VI_h	any	6± or −	$\rho = 0$, Robinson–Bertotti type symmetry	(4.15)	Many bifurcate horizons, no singularities
{0}	various	any	10	Minkowski, de Sitter and anti-de Sitter spaces		Many bifurcate horizons, no singularities

three families which were shown in Siklos (1978) to be the only vacuum spatially homogeneous models to admit horizons.

The notation and conventions of Siklos (1978) are followed throughout this paper, except that the function $\exp(-2\eta)$ is now called simply f.

2. The null tetrad formalism

The space–times under consideration are algebraically special, and are (at least locally) foliated by a family of homogeneous hypersurfaces $S(u)$, which are invariant under a three-parameter Lie group G_3 of Killing motions. As in Siklos (1978), a G_3-invariant null tetrad (Newman and Penrose 1962) is chosen as follows. l^a is a repeated principal null direction (PND) of the Weyl tensor (which means that the Weyl tensor components Ψ_0 and Ψ_1 (Newman and Penrose 1962) vanish (Pirani 1964)) and n^a is defined uniquely by

$$\sqrt{2}\tilde{u}^a = fl^a + n^a; \qquad l^a n_a = 1 \tag{2.1}$$

for any given $\eta(u)$, where $\tilde{u}^a$ is normal to $S(u)$. $(m^a + \bar{m}^a)$ and $\mathrm{i}(m^a - \bar{m}^a)$ are tangent to $S(u)$ and can also be chosen to be G_3-invariant. The invariant scalar f is positive when $\tilde{u}^a$ is time-like, negative when $\tilde{u}^a$ is space-like and vanishes on an isometry horizon when $\tilde{u}^a$ is null. The possibility that the only repeated PND lies in the horizon must also be borne in mind, although it does not arise here. The remaining freedom in the choice of tetrad is

$$l^a \to Al^a, \qquad n^a \to A^{-1}n^a; \qquad A = A(u) \tag{2.2}$$

$$m^a \to \exp(\mathrm{i}\theta)m^a; \qquad \theta = \theta(u). \tag{2.3}$$

The spin coefficients for the null tetrad are defined by

$$\begin{aligned} l_{a;b} &= \mathrm{Re}[(\gamma+\bar{\gamma})l_a l_b + (\epsilon+\bar{\epsilon})l_a n_b - 2(\alpha+\bar{\beta})l_a m_b - 2\bar{\tau} m_a l_b \\ &\quad + 2\bar{\sigma} m_a m_b + 2\rho \bar{m}_a m_b - 2\bar{\kappa} m_a n_b] \\ n_{a;b} &= \mathrm{Re}[-(\epsilon+\bar{\epsilon})n_a n_b - (\gamma+\bar{\gamma})n_a l_b + 2(\alpha+\bar{\beta})n_a m_b \\ &\quad + 2\pi m_a n_b - 2\lambda m_a m_b - 2\bar{\mu}\bar{m}_a m_b + 2\nu m_a l_b] \\ m_{a;b} &= (\bar{\beta}-\alpha)m_a m_b + (\bar{\alpha}-\beta)m_a \bar{m}_b + (\gamma-\bar{\gamma})m_a l_b + (\epsilon-\bar{\epsilon})m_a n_b - \bar{\mu} l_a m_b \\ &\quad + \rho n_a m_b - \bar{\lambda} l_a \bar{m}_b + \sigma n_a \bar{m}_b + \bar{\pi} l_a n_b - \tau n_a l_b + \bar{\nu} l_a l_b - \kappa n_a n_b. \end{aligned}$$

With this choice of tetrad, the spin coefficients and the Riemann tensor components are invariant (that is, functions of u only). Also

$$\delta\phi = (fD - \Delta)\phi = 0 \tag{2.4}$$

for any invariant function $\phi(u)$, where D, Δ and δ are the usual (Newman and Penrose 1962) derivative operators in the l^a, n^a and m^a directions respectively. When applied to the commutation relations for the null tetrad (Siklos 1978), equation (2.4) implies the following useful identities:

$$Df = -(\gamma + \bar{\gamma}) \tag{2.5}$$

$$0 = (\bar{\alpha} + \beta - \bar{\pi}) \tag{2.6}$$

$$0=-\bar{\nu}+(\tau-\bar{\alpha}-\beta)f \tag{2.7}$$

$$0=(\bar{\mu}-\mu)+(\bar{\rho}-\rho)f. \tag{2.8}$$

Here we have set

$$\kappa=\sigma=\epsilon=0 \tag{2.9}$$

using a generalisation of the Goldberg–Sachs theorem (Pirani 1964) and the freedom (2.2) and (2.3) under which

$$\epsilon\to A\epsilon-(\mathrm{i}/2)D\theta+(1/2)DA.$$

The shear and expansion of the normal congruence, and the quantities n^{AB} and a_B used to classify the different group types (Ellis and MacCallum 1969) are given in Siklos (1978). The criteria for the tetrad to be invariant under a class A group (Ellis and MacCallum 1969) are

$$\tau=2\beta \qquad \text{and} \qquad \mu+\rho f=0. \tag{2.10}$$

3. Integration of the field equations

When equations (2.4)–(2.9) hold, the Newman–Penrose equations (Newman and Penrose 1962) become

$$D\rho=\rho^2 \tag{3.1a}$$

$$D\tau=(\tau+\bar{\pi})\rho \tag{3.1c}$$

$$D\alpha=(2\pi-\bar{\beta})\rho \tag{3.1d}$$

$$D\beta=\bar{\rho}\beta \tag{3.1e}$$

$$D\gamma=\pi\bar{\pi}+2\tau\pi+\bar{\tau}\beta-\tau\bar{\beta}+\pi\beta-\overline{\pi\beta}+\Psi_2-\Lambda \tag{3.1f}$$

$$D\lambda=\rho\lambda+2\pi(\pi-\bar{\beta}) \tag{3.1g}$$

$$D\mu=\bar{\rho}\mu+2\beta\pi+\Psi_2+2\Lambda \tag{3.1h}$$

$$\Delta\lambda=-\lambda(\mu+\bar{\mu}+3\gamma-\bar{\gamma})+(4\pi-2\bar{\beta}-\bar{\tau})\nu-\Psi_4 \tag{3.1j}$$

$$0=\bar{\pi}\rho+(\rho-\bar{\rho})\tau \tag{3.1k}$$

$$0=\mu\rho+\gamma(\rho-\bar{\rho})+\pi\bar{\pi}-3\pi\beta-\overline{\beta\pi}+4\beta\bar{\beta}-\Psi_2+\Lambda \tag{3.1l}$$

$$0=(\rho-\bar{\rho})\nu+(2\mu-\bar{\mu})\pi+\lambda(\bar{\pi}-4\beta)-\Psi_3 \tag{3.1m}$$

$$-\Delta\mu=\mu^2+\lambda\bar{\lambda}+(\gamma+\bar{\gamma})\mu-\bar{\nu}\pi+(\tau-\bar{\pi}-2\beta)\nu \tag{3.1n}$$

$$-\Delta\beta=(\tau-\bar{\pi})\gamma+(\tau+\beta)\mu-\beta(\gamma-\bar{\gamma})+(\pi-\bar{\beta})\bar{\lambda} \tag{3.1o}$$

$$0=\bar{\lambda}\rho+(\tau+2\beta-\bar{\pi})\tau \tag{3.1p}$$

$$\Delta\rho=-\rho\bar{\mu}+(2\bar{\beta}-\pi-\bar{\tau})\tau+\rho(\gamma+\bar{\gamma})-\Psi_2-2\Lambda \tag{3.1q}$$

$$\Delta\alpha=\rho\nu-(\tau+\beta)\lambda+(\bar{\gamma}-\bar{\mu})(\pi-\bar{\beta})+\gamma(\bar{\beta}-\bar{\tau})-\Psi_3. \tag{3.1r}$$

The numbering of these equations follows that of Newman and Penrose (1962), and equations (b) and (i) are identically satisfied (the latter by virtue of equations (2.5)–

(2.9)). The only non-trivial Bianchi identities are

$$0 = \tau\Psi_2 \tag{3.2a}$$

$$D\Psi_2 = 3\rho\Psi_2 \tag{3.2b}$$

$$\Delta\Psi_2 = -3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 \tag{3.2c}$$

$$D\Psi_3 = 3\pi\Psi_2 + 2\rho\Psi_3 \tag{3.2d}$$

$$\Delta\Psi_3 = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 \tag{3.2e}$$

$$D\Psi_4 = -3\lambda\Psi_2 + 2(3\pi - \bar{\beta})\Psi_3 + \rho\Psi_4. \tag{3.2f}$$

From equations (3.1) and (3.2) one can find all the spin coefficients in terms of u by choosing D such that $D\phi = \mathrm{d}\phi/\mathrm{d}u$ for $\phi = \phi(u)$. The two cases $\rho = 0$ and $\rho \neq 0$ have to be considered separately. Equations (3.1a–f) and (3.2b) can be integrated directly, using equations (2.6) and (3.1k) to give

$$\rho = 0, \qquad \beta = b, \qquad \pi = \pi_0,$$
$$\alpha = \pi_0 - b, \qquad \tau = \tau_0, \qquad \Psi_2 = \Psi_{2_0}$$

and

$$\gamma = [2\tau_0\pi_0 + \pi_0\bar{\pi}_0 + b(\bar{\tau}_0 - \tau_0) + b(\pi_0 - \bar{\pi}_0) + \Psi_2 - \Lambda]u + \gamma_0 \tag{3.3}$$

or

$$\rho = -(u + \mathrm{i}a)^{-1}, \qquad \beta = b\bar{\rho}, \qquad \pi = \pi_0\rho^2, \qquad \alpha = \pi_0\rho^2 - b\bar{\rho},$$
$$\tau = \tau_0\bar{\rho}, \qquad \pi_0 = 2\mathrm{i}a\bar{\tau}_0, \qquad \Psi_2 = \Psi_{2_0}\rho^3$$

and

$$\gamma = \tau_0\bar{\tau}_0(\rho - \bar{\rho}) + b(\bar{\tau}_0\rho - \tau_0\bar{\rho}) + \Lambda/\rho + \Psi_2/2\rho + \gamma_0. \tag{3.4}$$

The quantities a, b and those with the suffix 0 are constants, and the transformations $u \to u + \text{constant}$ and (2.3) with θ constant have been used to make a and b real.

Equations (3.1p) and (3.1l), which correspond to the Jacobi identities for the three invariant space-like vector fields, give

$$(\tau_0 + 2b - \pi_0)\tau_0 = 0 \qquad \text{and} \qquad \pi_0\bar{\pi}_0 - 3b\pi_0 - b\bar{\pi}_0 + 4b^2 + \Lambda - \Psi_2 = 0 \tag{3.5}$$

when $\rho = 0$, and

$$\begin{aligned}\mu &= -4b^2\bar{\rho} + 2b\bar{\tau}_0(\rho - \bar{\rho}) + \gamma_0(\bar{\rho}/\rho - 1) + \Psi_{2_0}(\rho + \bar{\rho})/2 + \Lambda(\bar{\rho}/\rho^2 - 2/\rho)\\ \lambda &= -\bar{\tau}_0\rho^2(\bar{\tau}_0 + 2b - \pi_0\rho)/\bar{\rho}\end{aligned} \tag{3.6}$$

when $\rho \neq 0$. Equations (3.1g) and (3.1h) are identically satisfied when $\rho \neq 0$, but yield

$$\mu = 2b\pi_0 + \Psi_2 + 2\Lambda + \mu_0, \qquad \lambda = 2\pi_0(\pi_0 - b) + \lambda_0 \tag{3.7}$$

when $\rho = 0$. In both cases, ν is given by equation (2.7).

The method of solving the remaining equations is to obtain τ in terms of β from the Bianchi identities for each of the algebraically special Petrov–Pirani types separately, and then use equation (3.1). f is derived from equation (2.8) if $\rho \neq \bar{\rho}$, and from equation (2.5) otherwise. The calculations are straightforward, though often lengthy. I shall not give any further details here. When the spin coefficients are known, the group type is found by substitution into the expressions for n^{BA} and a_{B} given in Siklos (1978). Next, the null tetrad is expressed in terms of $\{X_i\}$ ($i = 1, 2, 3$), a u-independent triad invariant under the appropriate group type, as follows:

$$\Delta = f\frac{\partial}{\partial u} + \sum_i A^i X_i, \qquad D = 2\frac{\partial}{\partial u} - f^{-1}\Delta, \qquad \delta = \sum_i P^i X_i. \tag{3.8}$$

Here A^i and P^i are functions of u only, which can be found from the spin coefficients by means of the commutator relations. Finally, the metric is calculated using $g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$. Equation (3.8) gives the metric in comoving coordinates, but in some cases, notably when the group is of type VI_h with $h = -\frac{1}{9}$, this leads to elliptic functions. These can be avoided by using a tilted coordinate system with $D = \partial/\partial u$ and the other tetrad derivatives as in equation (3.8).

4. The solutions

The solutions are given in §§ 4.1–4.4 below. Listed in each case are: the spin coefficient ρ; the constants Λ (which is minus one sixth of the usual cosmological constant (Hawking and Ellis 1973)), b, π_0, τ_0, γ_0, and, if $\rho = 0$, μ_0 and λ_0; the quantity f; and the metric. All the remaining spin coefficients can then be found from equations (2.6), (2.7) and (3.3)–(3.7). The group type, the number of independent isometries, and other information are given when appropriate.

4.1. Homogeneous plane waves

The only non-zero spin coefficients in this solution are constant:

$$\gamma = \tfrac{1}{2}(m + \mathrm{i}k), \qquad \mu = -am, \qquad \lambda = 1 \tag{4.1}$$

which must satisfy

$$0 = a^2 + a + 1/m^2 \tag{4.2}$$

and $f = -2mu$.

The Weyl tensor is of Petrov–Pirani type {4}, and represents a homogeneous plane wave (Ehlers and Kundt 1962). The metric in standard plane-wave form is

$$\mathrm{d}s^2 = -2\,\mathrm{d}\theta\,\mathrm{d}\bar{\theta} + 2\,\mathrm{d}p\,\mathrm{d}q + 2H\,\mathrm{d}p^2 \tag{4.3}$$

with

$$H = \mathrm{Re}\{C\theta^2 p^{-2(1-\mathrm{i}\kappa)}\} \tag{4.4}$$

and

$$\kappa = \frac{k}{m}; \qquad C = \frac{1}{m}\left(\frac{4k^2}{m^4} + \frac{(2a+1)^2}{m^2}\right)^{1/2}. \tag{4.5}$$

The group type is VI_h, VII_h or IV, depending on whether $(k^2 - 1)$ is negative, positive or zero respectively. In the first two cases

$$h = a(1-a)/(1-k^2). \tag{4.6}$$

The spatially homogeneous parts of the space–time are described by the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{t^2\,\mathrm{d}z^2}{4b^2} - v^{-2a}\Omega_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \qquad (i, j = 1, 2) \tag{4.7}$$

where $b = |1-k^2|/2m$ and $v = t\exp(z/2b)$, $u = t\exp(-z/2b)$. The appropriate forms for Ω_{ij} for the three different group types are

$$\Omega = \begin{pmatrix} 1 & -\dfrac{1}{m}\ln v \\ -\dfrac{1}{m}\ln v & \left(\dfrac{\ln v}{m}\right)^2 + 1 \end{pmatrix} \qquad \text{for type IV } (k=1)$$

$$\Omega = \begin{pmatrix} k+\cos(2b\ln v) & -\sin(2b\ln v) \\ -\sin(2b\ln v) & k-\cos(2b\ln v) \end{pmatrix} \qquad \text{for type VII}_h\ (k>1)$$

$$\Omega = \begin{pmatrix} k+\cosh(2b\ln v) & -\sinh(2b\ln v) \\ -\sinh(2b\ln v) & k-\cosh(2b\ln v) \end{pmatrix} \qquad \text{for type VI}_h\ (k<1).$$

The type IV metric has been given by Harvey and Tsoubelis (1977), the type VI_h metric by Collins (1972), and the type VII_h metric by Lukash (1975). Note that equations (4.2) and (4.5) imply that the solution (4.3) admits a three-parameter group acting transitively on space-like hypersurfaces (i.e. is equivalent to (4.7)) only when either $0 \leqslant C \leqslant \kappa^2 + \frac{1}{4}$ (if $\kappa \leqslant \frac{1}{2}$) or $0 \leqslant C \leqslant \kappa$ (if $\kappa \geqslant \frac{1}{2}$).

The constants κ and C in equation (4.4) determine the space–time uniquely: two space–times with metrics of the form (4.3) are isometric only if these constants agree. It is apparent from equations (4.5), however, that the group invariant constants a, b and k are not uniquely defined by κ and C, so each plane-wave space–time admits various foliations corresponding to different values of a, b and k. It can be deduced from the relation $\kappa^2 - C^2 = (k^2-1)(1-4/m^2)$, and the inequality $|m| > 2$ (which follows from equation (4.2)), that plane waves with $\kappa > C$ admit only type VII_h (i.e. $k>1$) groups, while those with $\kappa < C$ admit only type VI_h groups. When $\kappa = C$, the space–time allows type IV groups and also either type VII_h (if $C > \frac{1}{2}$) or type VI_h (if $C < \frac{1}{2}$); this arises from the possibility $m^2 = 4$.

A careful examination of equations (4.5) and (4.2) reveals that if $\kappa > C$ there are for each (κ, C) two distinct foliations corresponding to type VII_h groups with different values of h (these values can be found from equation (4.6)) and if $\kappa < C$ there are four possible type VI_h groups. In the case $\kappa = C$ either a type VI_h group (if $C < \frac{1}{2}$) or a type VII_h group (if $C > \frac{1}{2}$) is replaced by a group of type IV. Clearly the invariant hypersurfaces for the different group types cannot coincide unless there is a fourth Killing motion acting in the hypersurfaces; but this cannot happen in these cases because there would then be an isotropy acting on space-like hypersurfaces and the Weyl tensor would therefore have to be Petrov–Pirani type {2, 2} or {0}.

4.2. Type $VI_{-1/9}$ solutions

This is the exceptional group type (see Ellis and MacCallum 1969, Siklos 1978) for which the constraint equations are degenerate. There are six such solutions in the class under consideration. Their metrics are rather similar:

$$ds^2 = 2\omega^2[du - 5u\omega^1 + k\omega^2] - (5/2b^2)(\omega^{12} + \omega^{32}) \tag{4.8}$$

$$ds^2 = 2\omega^3[du + 4u\omega^1 - \omega^3 + 2\omega^2/b] - (1/2b^2)(\omega^{12} + \omega^{22}) \tag{4.9}$$

$$ds^2 = 2\omega^2[du + u\omega^1 - (6 + \Lambda u^2)\omega^2] - 2u^2(\omega^{12} + \omega^{32}) \tag{4.10}$$

$$ds^2 = 2\omega^3[du + 2u\omega^1 + 4b^2(8u^2 - \tfrac{1}{2})\omega^3 + 2\omega^2] - (1/8b^2)(\omega^{12} + \omega^{22}) \tag{4.11}$$

$$ds^2 = 2(\omega^3 - \tfrac{1}{3}\omega^2)[du + 2(u\omega^1 + \omega^2) + (u^2-1)/2(\omega^3 - \tfrac{1}{3}\omega^2)] - (u^2+1)/2(\omega^{12} + \omega^{22}) \tag{4.12}$$

$$ds^2 = 2(\omega^2 + \tfrac{4}{3}\omega^3)[du - (u\omega^1 + \omega^3) + (13u^2+17)/32(\omega^2 + \tfrac{4}{3}\omega^3)] - 2(u^2+1)/(\omega^{22} + \omega^{32}). \tag{4.13}$$

The group-invariant one-forms ω^i can be taken to be $\omega^1 = dz$, $\omega^2 = \exp(-z)\,dx$ and $\omega^3 = \exp(2z)\,dy$. The coordinate u may differ by a constant multiple from the one defined by l^a. The spin coefficients and other properties of these metrics are listed in table 2.

The solutions (4.8) and (4.9) have been given by Kaigorodov (1963) while (4.12) is Leroy's (1970) solution.† The others are believed to be new, though when $\Lambda = 0$ in equation (4.10), one has the Robinson–Trautman solution (Robinson and Trautman 1962) which was also studied by Collinson and French (1967). The solution (4.13) is closely related to Leroy's solution, but is of Petrov–Pirani type {31} instead of {4}. The final forms of both these metrics are due to MacCallum.

4.3. Type {22} solutions

The remaining space–times in the class under consideration are all of Petrov–Pirani type {22}, and are related to NUT–de Sitter space (Ruban 1972, Plebanski 1975, Siklos 1977). The Schwarzschild–de Sitter solution also comes into this category, but is not covered by this method because it has no simply transitive three-parameter isometry group.

When $\rho \neq 0$, the metrics can be written in the form

$$ds^2 = -f^{-1}\,du^2 + f(d\psi + f(\theta)\,d\phi)^2 + (t^2 + l^2)(d\theta^2 + g(\theta)\,d\phi^2) \tag{4.14}$$

where $f(\theta) = 2lb^{-1/2}\sin k^{1/2}\theta$, $g(\theta) = \cos^2 k^{1/2}\theta$ and

$$f = [-6k(u^2 - l^2)l^2 + 2mu + \Lambda(u^4 + 6l^2u^2 - 3l^4)](u^2 + l^2)^{-1}. \tag{4.15}$$

The spin coefficients and the various parameters occurring in equation (4.14) are explained in table 3.

When $\rho = 0$, the three possible solutions are each isometric to

$$(12\Lambda)\,ds^2 = (\cosh^2 x\,dt^2 - dx^2) - (dy^2 + \cosh^2 y\,dz^2). \tag{4.16}$$

This line element describes a manifold which is a product of two two-spaces of constant negative curvature. If one writes down the six Killing vector fields for (4.16), one can easily see the combinations which give rise to the homogeneous hypersurfaces described in table 3.

4.4 Conformally flat solutions

This is the hardest case, even though the solutions are just Minkowski, de Sitter and anti-de Sitter spaces. The difficulties are due partly to the Bianchi identities being trivial, and partly to the fact that there is no easy choice of l_a. (One cannot simply choose $l_{a;b} = 0$, because this vector field may not be G_3-invariant.) It is, however,

† I am grateful to M A H MacCallum for bringing this solution to my attention.

Table 2. The type $VI_{-1/9}$ metrics. The spin coefficients can be reconstructed using equations (3.3)–(3.7).

Metric eqn.	Petrov type	Λ	No of Killing vectors	ρ	τ_0	π_0	γ_0	μ_0	λ_0	f
(4.8)	{4}	$-16b^2$	5	0	$4b$	$6b$	0	0	0	$5(k-b^2u^2)$
(4.9)	{31}	$-b^2$	4	0	b	$3b$	$-\frac{\mathrm{i}}{2}$	0	i	$-\left(16b^2u^2+\frac{1}{18b^2}\right)$
(4.10)	{31}	Λ	3	$-\frac{1}{u}$	b	0	0	—	—	$(5+u^2)$
(4.11)	{211}	$-\frac{4}{3}b^2$	3	0	0	$4b$	i	0	$-2\mathrm{i}$	$-\left(20b^2u^2+\frac{1}{18b^2}\right)$
(4.12)	{4}	$-2b^2$	3	$-(u+\mathrm{i})^{-1}$	$4b$	—	0	—	—	$-2b^2(u^2+7)$
(4.13)	{31}	$-\frac{13}{8}b^2$	3	$-(u+\mathrm{i})^{-1}$	b	—	0	—	—	$-b^2\left(\frac{13u^2+25}{8}\right)$

Table 3. The type {22} solutions.

Λ	ρ	γ_0	λ_0	μ_0	π_0	τ_0	β	Group type	f	Metric eqn.
$-\frac{4}{3}b^2$	0	0	0	0	0	0	b	III	$-4b^2-k$	(4.16)
$-\frac{4}{3}a^2$	0	$-\mathrm{i}$	i	0	a	0	0	VIII	$-(2a^2u^2+\frac{1}{2}a^2)$	(4.16)
$-\frac{4}{3}(a-b)^2$	0	0	0	0	$a+b$	0	b	$VI_h(h=-b^2/a^2)$	$-2(a^2+b^2)u^2$	(4.16)
Λ	$-(u+\mathrm{i}l)^{-1}$	$3\mathrm{i}lk$	—	—	0	0	0	I $(l=0)$ II $(k=0)$ VIII $(k=-1)$ IX $(k=1)$	see eqn. (4.15)	(4.14)
Λ	$-(u+\mathrm{i}l)^{-1}$	0	—	—	0	0	$\bar{\rho}/2$	III	(4.15) with $k=-1$	(4.14)

always possible to choose $\kappa=\sigma=\epsilon=0$, in accordance with equation (2.9). The different solutions represent all non-null homogeneous hypersurfaces, except those corresponding to $m=0$ Schwarzschild, and are given in table 4.

5. Global properties

This section deals with some of the global properties of the solutions derived in § 4, in particular their horizons and singularities. Using the null tetrad formalism, this amounts to little more than a cursory examination of the Weyl tensor components, certain spin coefficients, and the function f.

It can easily be shown that each non-null homogeneous hypersurface is (generalised affine parameter (Hawking and Ellis 1973)) complete, and that null homogeneous hypersurfaces are complete if and only if their surface gravity (Ellis and King 1974, Boyer 1969) vanishes. In the space-times under consideration, the hypersurfaces become null when $f=0$, and are therefore complete if $(\gamma+\bar{\gamma})=0$ on the horizon.

When the horizon is incomplete, it will generally be singular at both 'ends'. A singularity occurs when components of the Weyl tensor become unbounded in a frame which is parallelly propagated (PP) along some curve, in this case the null generators of the horizon. Because the horizon is a surface of homogeneity, Weyl tensor components are constant in a group-invariant tetrad. Components in the PP frame, $\tilde{\Psi}_i$, are related to these constant components Ψ_i by

$$\tilde{\Psi}_i=s^{i-2}\Psi_i$$

where s is an affine parameter along the null geodesics which rule the horizon. Weyl tensor components will therefore diverge at $s=0$ and $s=\infty$, unless some of the Weyl tensor components vanish in the invariant tetrad (in fact, unless the Weyl tensor is algebraically special on the horizon).

Completeness of the congruence normal to the homogeneous hypersurfaces can be tested using the affine parameter, t, along these geodesics:

$$t=\frac{1}{\sqrt{2}}\int^u |f^{-1/2}|\,\mathrm{d}u. \tag{5.1}$$

In the frame parallel along the normal congruence, the Weyl tensor components are given by

$$f^{(2-i)/2}\Psi_i \tag{5.2}$$

if $\pi=0$. Equation (5.2) holds approximately provided π is bounded (which is always the case here) so it can be used to determine whether or not the normal congruence encounters a singularity.

The causal structures of the more interesting solutions are represented in figures 1–5, which are conformal diagrams spanned by l^a and n^a. This is not strictly possible unless l^a and n^a are involutive, that is, unless $(\tau+\bar{\pi})=0$. If m^a and $\bar{m}^a$ are also involutive ($\rho=\bar{\rho}$, $\mu=\bar{\mu}$) the metric can be written as the direct sum of two two-dimensional metrics, one of which has the conformal structure pictured. If m^a and $\bar{m}^a$ are non-involutive, the manifold may be regarded as a fibre bundle for which the vertical subspace tangent to the fibre is shown in the diagram (cf Hawking and Ellis 1973 p 173). When $(\tau+\bar{\pi})\neq 0$, the diagram is schematic.

Table 4. Conformally flat solutions (see § 4.4). The asterisk by some of the Λ indicates that $\Lambda = 0$ gives a solution not given elsewhere in the table.

Λ	ρ	γ_0	λ_0	μ_0	π_0	τ_0	β	Group type	f
0	0	1	1	1	0	0	0	III	$-2u$
0	0	$1+ik$	0	0	0	0	0	I or VII_0	$-2u$
0	0	0	0	0	$2b$	0	b	III	$-4b^2u^2$
0	0	$1+ik$	0	-2	0	0	0	V or VII_h	$-2u$
$-a^2$	0	0	1	-1	a	a	0	III	$-4a^2u^2+a^{-2}$
$-(a-b)^2$	0	0	0	0	$a+b$	$a-b$	b	VI_h, $h=-\left(\dfrac{a+b}{a-3b}\right)^2$	$-4a^2u^2$
$-b^2$	0	0	1	3	b	$-b$	b	$\mathrm{VI}_{-1/9}$	$-1/b^2$
$-b^2$	0	0	0	0	$-b$	$-2b$	b	V	$-4b^2u^2+k$
Λ^*	$-u^{-1}$	m	—	—	0	-2	u^{-1}	V	$\Lambda u^2-2mu-4$
Λ	$-u^{-1}$	$m+ik$	—	—	0	0	0	I or VII_0	Λu^2-2mu
Λ^*	$-u^{-1}$	0	—	—	0	0	u^{-1}	III	Λu^2+4
Λ	$-u^{-1}$	0	—	—	0	1	0	III	Λu^2-1
Λ	$-(u+ia)^{-1}$	$-2ia\Lambda$	—	—	0	0	0	II, VIII, IX	$\Lambda(u^2+a^2)$
$-a^{-2}$	$-(u+ia)^{-1}$	0	—	—	0	0	$\bar{\rho}$	III	$-(u^2/a^2+1)$

5.1. The plane wave solutions

The solution given by equation (4.1) has $f = -2mu$, so the surfaces of homogeneity are space-like for $u < 0$, and time-like for $u > 0$. There is a horizon at $u = 0$. The $t - z$ part of the metric (4.7) is conformal to half of Minkowski space (figure 1). Since the space–time is homogeneous (it has a simply transitive G_4), the horizon does not define a unique hypersurface: it could be chosen to occur at any value of u.

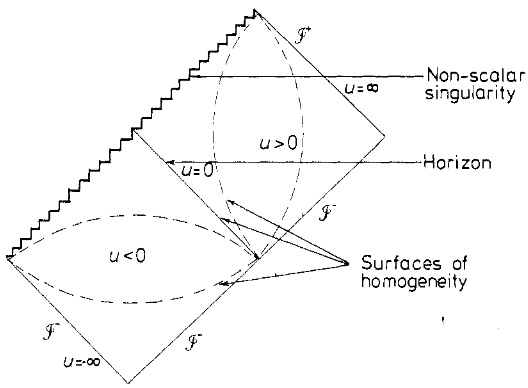

Figure 1. The conformal diagram for the maximally extended t–z plane of the plane-wave space–time described by the metric (4.3).

The normal congruence runs into a singularity as $u \to 0$, as can be seen from equation (5.2). This singularity has the causal structure (Geroch *et al* 1972) of a null curve.

The metric (4.7) satisfies not only the exact Einstein vacuum equations, but also the linearised ones. The type VII_h case is particularly interesting because it is precisely the growing mode found by Collins and Hawking (1973) in their investigation of spatially homogeneous perturbations of the Friedmann universe.

5.2. The type $VI_{-1/9}$ solutions

The two solutions with $\rho \neq \bar{\rho}$ (metrics (4.12) and (4.13)) are both non-singular and both have $f < 0$ everywhere. The space–times are therefore geodesically complete, and are foliated by time-like homogeneous hypersurfaces. The same applies to solutions (4.9) and (4.11) and also to (4.8) when $k \leq 0$. In all these cases (in fact, whenever there is a negative Λ term) future null infinity is space-like.

When k is positive in equation (4.8), there are horizons at $u = \pm(k/b)^{1/2}$ and the surfaces of homogeneity are space-like between the two horizons. The l^a congruence does not encounter any singularities, but the normal congruence becomes singular at both horizons (see equation (5.2)). The conformal diagram is therefore as shown in figure 2; it is an example of the 'whimper–whimper' configuration of Ellis and King (1974). As in the case of plane waves, the horizons are not unique, because the space–time is homogeneous.

In the generalised Collinson–French solution (4.10), both Ψ_3 and Ψ_4 diverge as $u \to 0$, and these components also diverge in a tetrad which is parallel along the n^a congruence. When $\Lambda \geq 0$ there are no horizons and the homogeneous hypersurfaces are space-like. The causal structure is that of figure 3 when $\Lambda = 0$, but has a space-like

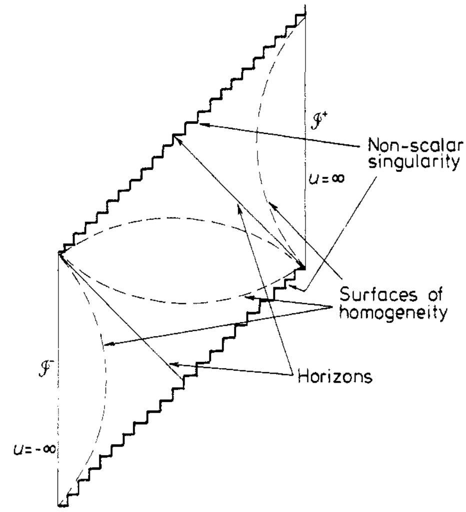

Figure 2. The 'whimper-whimper' space-time described by the spin coefficients of (4.8) when $k>0$.

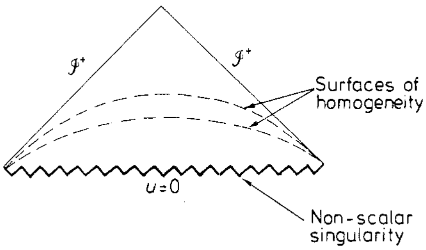

Figure 3. A conformal representation of the Collinson–French space-time given by equation (4.10) with $\Lambda=0$.

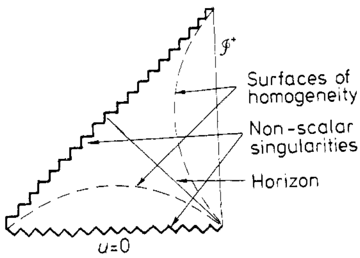

Figure 4. A conformal representation of the generalised ($\Lambda<0$) Collinson–French space-time as given by equation (4.10).

null infinity when $\Lambda>0$. The $\Lambda=0$ solution is of particular interest because it is a vacuum solution with a non-scalar singularity but no horizon. This configuration only occurs for type $\mathrm{VI}_{-1/9}$ groups (Siklos 1980). When $\Lambda<0$, there are horizons at $u=\pm(-5/\Lambda)^{1/2}$; these occur in disjoint regions which are separated by the singularity at $u=a$. Each has the structure shown in figure 4.

5.3. The type {22} solutions

These fall into the Schrödinger-separable class investigated by Carter (1967, 1968). When $\rho\neq\bar{\rho}$, the only non-zero component of the Weyl tensor in the canonical tetrad, Ψ_2, cannot diverge, so the solutions have no curvature singularities ($\pi=\tau=0$ in this tetrad). However, when the group is type IX one encounters Taub–NUT-like behaviour (Hawking and Ellis 1973). This is a consequence of the group being compact, and does not occur for types VIII and II (Siklos 1977, Miller 1977). The $\rho=\bar{\rho}$ solution is LRS Kasner with a cosmological constant (Ellis and MacCallum 1969), and has the usual Kasner singularity at $u=0$. The $\rho=0$ solution is non-singular.

The conformal diagram for the non-singular solutions can easily be constructed from the following properties of f. Each real root of $f=0$ determines a horizon, and if two or more roots are equal, the horizon is degenerate. If an even number of roots are equal, the surfaces of homogeneity are space-like (or time-like) on both sides of the horizon. Otherwise the hypersurfaces go from space-like to time-like. Each non-degenerate horizon is bifurcate (Boyer 1969), and each degenerate horizon is non-bifurcate and extends to infinity at both ends. If $\Lambda>0$, the hypersurfaces 'near' null infinity are space-like, and if $\Lambda<0$, they are time-like. In fact for solutions with given Λ/k (see table 2) the conformal diagrams for $\Lambda>0$ and $\Lambda<0$ are related by a rotation of $\pi/2$. The conformal diagram for a solution with two non-degenerate, and one singly degenerate horizon, and $\Lambda<0$ is shown in figure 5. The different possibilities for the roots of $f=0$ are shown in table 5, for the cases $\rho\Lambda\neq 0$.

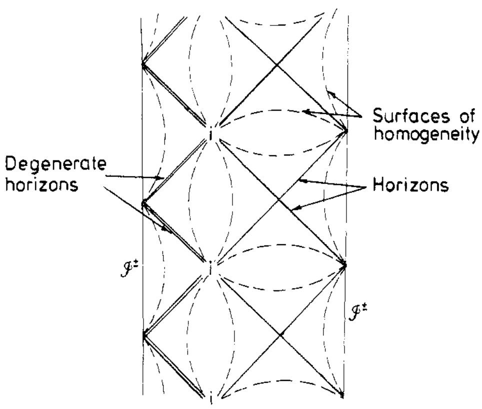

Figure 5. The maximally extended type {22} space–time represented by the metric (4.15) in the case when the equation $f=0$ has four real roots, two of which are equal, and $\Lambda>0$. The points labelled 'i' are at infinity.

Table 5. The real roots of $f=0$, where f is given by equation (4.15) and $\Lambda\neq 0$. Here $\Delta\equiv I^3-27J^2$ where $I=3(k/\Lambda)^2$ and $J=[(k/\Lambda)^2-8(k/\Lambda)+4](k/\Lambda-1)-m^2/4\Lambda^2$.

Real roots of $f=0$	$m\neq 0$	$m=0$
4 unequal roots	$\Delta>0,\ k/\Lambda>2$	$k/\Lambda>2$
4 roots, one equal pair	$\Delta=0,\ k/\Lambda>2$	—
4 roots, two equal pairs	—	$k/\Lambda=2$
4 equal roots	—	$l^2=0$
2 unequal roots	$\Delta<0$	$k/\Lambda<\frac{1}{2}$
2 equal roots	$\Delta=0,\ k/\Lambda\leq 2$	$k/\Lambda=\frac{1}{2}$
no roots	$\Delta>0,\ k/\Lambda\leq 2$	$\frac{1}{2}<k/\Lambda<2$

5.4. *Conformally flat solutions*

These are of course non-singular. The causal structure of the homogeneous hypersurfaces can be determined using the method of the preceding paragraph, which applies to this case without modification.

5.5. *Other algebraically special Einstein spaces*

The solutions described in §§ 5.1–5.4 do not include those special cases mentioned in the introduction. When the only repeated principal null congruences of the Weyl tensor lie in the homogeneous hypersurfaces, the space–time is simply foliated by time-like hypersurfaces. When the homogeneous hypersurfaces are everywhere null, they must also be complete, so the space–time is foliated by null hypersurfaces. Finally, the case of Kantowski–Sachs symmetry (Kantowski and Sachs 1966), when there is no simply transitive three-parameter group, has been considered by Collins (1977).

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References

Boyer R H 1969 *Proc. R. Soc. Lond.* **311** 245
Carter B 1967 *PhD Thesis* University of Cambridge
—— 1968 *Commun. Math. Phys.* **10** 280
Collins C B 1972 *PhD Thesis* University of Cambridge
—— 1977 *J. Math. Phys.* **18** 2115
Collins C B and Hawking S W 1973 *Astrophys. J.* **180** 317
Collinson C D and French D C 1967 *J. Math. Phys.* **8** 701
Ehlers J and Kundt W 1962 *Gravitation: An Introduction to Current Research* ed L Witten (New York: Wiley)
Ellis G F R and King A R 1974 *Commun. Math. Phys.* **38** 119
Ellis G F R and MacCallum M A H 1969 *Commun. Math. Phys.* **12** 108
Ellis G F R and Schmidt B G 1977 *Gen. Rel. Grav.* **8** 129
Geroch R P, Kronheimer E H and Penrose R 1972 *Proc. R. Soc.* **327**A 545
Harvey A and Tsoubelis D 1977 *Phys. Rev.* **D15** 2734
Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* (Cambridge: CUP)
Kaigorodov V R 1963 *Sov. Phys. Dokl.* **7** 893
Kantowski R and Sachs R K 1966 *J. Math. Phys.* **7** 443
Leroy J 1970 *C. R. Acad. Sci., Paris* A**270** 1078
Lukash V N 1975 *Sov. Phys.-JETP* **40** 792
Miller J G 1977 *Commun. Math. Phys.* **52** 1
Newman E and Penrose R 1962 *J. Math. Phys.* **3** 566
Pirani F A E 1964 *Lectures on General Relativity* ed H Bondi, F A E Pirani and A Trautman (Englewood Cliffs, NJ: Prentice Hall)
Plebanski J F 1975 *Ann. Phys.* **90** 196
Robinson D and Trautman A 1962 *Proc. R. Soc. Lond.* A**265** 463
Ruban V A 1972 *Dokl. Acad. Nauk. CCCP* **204** 1085
Siklos S T C 1977 *Phys. Lett.* **59**A 173
—— 1978 *Commun. Math. Phys.* **58** 255
—— 1980 submitted to *Gen. Rel. Grav.* (to be published)